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A Formula for Kergin Interpolation in R^k

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In this note we give a constructive proof of the existence of the map introduced by Kergin (*J. Approximation Theory* **29** (1980), 278–293).

Let S^n denote the n-simplex

$$\left\{(v_0, \ldots, v_n) : v_j \geqslant 0, \sum_{j=0}^{n} v_j = 1\right\}.$$

Given vectors $x^0, \ldots, x^n \in R^k$, consider the linear functional

$$\int_{[x^0, \ldots, x^n]} f = \int_{S^n} f(v_0 x^0 + \cdots + v_n x^n)\, dv_1 \cdots dv_n .$$

In particular, $\int_{[x^0]} f = f(x^0)$, while $\int_{[x^0, x^1]} f$ is the line integral of f over the segment $[x^0, x^1]$ and so on. We denote the directional derivative of f in the direction $y \in R^k$ by $D_y f$. Finally, we introduce a map $\pi_m : C^m(R^k) \to P_m(R^k)$, polynomials of total degree $\leqslant m$, by setting

$$\pi_m f(x) = \int_{[x^0, \ldots, x^m]} D_{x-x^0} \cdots D_{x-x^{m-1}} f.$$

This map is *independent* of the order of the points $x^0, \ldots, x^m$. To see this we let $f(x) = g(\lambda \cdot x)$, $g \in C^m(R^1)$, $\lambda \in R^k$ then

$$\pi_m f(x) = \lambda \cdot (x - x^0) \cdots \lambda \cdot (x - x^m) \int_{[\lambda \cdot x^0, \ldots, \lambda \cdot x^m]} g^{(m)},$$

which by the Hermite–Gennochi formula equals

$$\lambda \cdot (x - x^0) \cdots \lambda \cdot (x - x^m)[\lambda \cdot x^0, \ldots, \lambda \cdot x^m]\, g,$$

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where the expression $[x_0,\dots,x_m]\,g$ denotes the divided difference of g at $x_0,\dots,x_m$.

Now it is well known that the divided difference is a symmetric function of its arguments and, since the functions $g(\lambda \cdot x)$, $\lambda \in R^k$, $g \in C^m(R^1)$ are dense in $C^m(R^k)$, the symmetry of $\pi_m f$ in its arguments follows.

When we enumerate the maps $\pi_0, \pi_1, \dots, \pi_n$ based on $\{x^0,\dots,x^n\}$, we will always assume they are constructed from nested sets of the form $\{x^0\} \subseteq \{x^0, x^1\} \subseteq \cdots \subseteq \{x^0,\dots,x^n\}$, respectively, In this case we have

THEOREM 1.

$$\begin{aligned} \pi_r\pi_s &= 0, & r &\neq s \\ &= \pi_r, & r &= s. \end{aligned}$$

Proof. Again, for $f(x) = g(\lambda \cdot x)$ we have $\pi_s f(x) = \lambda \cdot (x - x^0) \cdots \lambda \cdot (x - x^{s-1})\ [\lambda \cdot x^0,\dots, \lambda \cdot x^{s-1}]\,g$. Now, if $r > s$, $\pi_r\pi_s f$ is zero because π_r annihilates polynomials of degree $<r$. On the other hand, for $r < s$ we see that $\pi_s f(x) = p(\lambda \cdot x)[\lambda \cdot x^0,\dots, \lambda \cdot x^{s-1}]\,g$, where $p(t) = (t - \lambda \cdot x^0) \cdots (t - \lambda \cdot x^{s-1})$. Thus $\pi_r\pi_s f(x) = \lambda \cdot (x - x^0) \cdots \lambda \cdot (x - x^{r-1})[\lambda \cdot x^0,\dots, \lambda \cdot x^{s-1}]\,g[\lambda \cdot x^0,\dots, \lambda \cdot x^r]\,p = 0$, and finally for $r = s$ we have $[\lambda \cdot x^0,\dots, \lambda \cdot x^r]\,p = 1$, which gives $\pi_r\pi_s f = \lambda \cdot (x - x^0) \cdots \lambda \cdot (x - x^{r-1}) \times [\lambda \cdot x^0,\dots, \lambda \cdot x^{s-1}]\,g = \pi_r f$ and proves the theorem.

Thus we see that π_r is a projection of $C^r(R^k)$ onto $P_r(R^k)$. The linear functionals which determine this projector are easily determined. In fact, $\pi_r f = 0$ iff $\int_{[x^0,\dots,x^r]} q(D) f = 0$ for all constant coefficient homogeneous differential operators of order r. This is again easily seen by letting $d^r f_{(P)}(v^0,\dots, v^{r-1}) =^{\text{def}} (D_{v^0} \cdots D_{v^{r-1}} f)(P)$, where $d^r f_{(x)} \in S^r L(R^n, R)$, i.e., it is a symmetric mapping from $\mathbb{R}^k \oplus \cdots \oplus_{r\text{-times}} \mathbb{R}^k$ into $\mathbb{R}$, linear in each of its r variables $v_j \in \mathbb{R}^k$. The partial derivatives $(D^\alpha f)(x)$ of order $|\alpha| = r$ are the coordinates of $d^r f_{(x)}$ in the vector space $S^r L(\mathbb{R}^k, \mathbb{R})$. Therefore $\pi_r f(x) \equiv 0$ iff $\int_{[x^0,\dots,x^r]} D^\alpha f = 0$ for all $|\alpha| = r$. Equivalently, $\pi_r f(x) \equiv 0$ iff $\int_{[x^0,\dots,x^r]} q(D) f = 0$ for all constant coefficient homogeneous differential operators of order r.

For any nested ordering of the points $\{x^0,\dots, x^n\}$ we define

$$T = \sum_{r=0}^{n+1} \pi_r .$$

Let us first observe that T is *independent* of the ordering chosen. Again we may see this by observing for $f(x) = g(\lambda \cdot x)$

$$Tf = \sum_{r=0}^{n+1} \lambda \cdot (x - x^0) \cdots \lambda \cdot (x - x^{r-1})[\lambda \cdot x^0,\dots, \lambda \cdot x^{r-1}]\,g.$$

We recognize this sum as the Newton form for the polynomial of degree $\leqslant n$ which interpolates g at $\lambda \cdot x^0,\ldots, \lambda \cdot x^n$. Thus T is independent of the order chosen for the points $x^0,\ldots, x^n$.

THEOREM 2. *T is Kergin interpolation on R^k.*

Proof. Let J be *any* subset of $\{0, 1,\ldots, n\}$ with $l+1$ integers. We choose an ordering of $\{x^0,\ldots, x^n\}$ so that $\{x^j \mid j \in J\}$ is $\{x^0,\ldots, x^l\}$. Then $\pi_l(Tf - f) = \pi_l f - \pi_l f = 0$. Hence, for any homogeneous differential operator q of order l, we have

$$\int_{[x^0,\ldots,x^l]} q(D)(Tf - f) = 0.$$